Stable Steiner Trees with Sparse Steiner Components

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Abstract

We study the metric Steiner tree problem under Bilu–Linial γ-stability, extending and sharpening known structural and algorithmic results for stable instances. We first show that in metrics of bounded doubling dimension, sufficiently stable instances have optimal solutions containing no Steiner points, so the optimum is exactly the minimum spanning tree on the terminals. In Euclidean space, we improve the no-Steiner-point stability threshold, and in the planar case we identify a lower stability regime in which Steiner points may occur but the optimum must be quasi-bipartite. Motivated by this structure, we give a polynomial-time algorithm for quasi-bipartite optima whenever $\gamma > (1+\sqrt{1+4\sqrt{2}})/2 \approx 1.7900$, despite the NP-hardness of the quasi-bipartite Steiner tree problem in general. Finally, we extend the algorithm to broader stable instances whose optimal Steiner components are sparse, with guarantees depending on the maximum number of Steiner neighbors of any Steiner vertex.

1 Introduction and previous work

Finding a minimum-weight Steiner tree is one of Karp's original 21 NP-hard problems. Freitag et al. [2022] initiated the study of finding minimum-weight Steiner trees under the assumption of γ-stability under multiplicative perturbations. This fits into the general context of Bilu–Linial stability [2012], a paradigm for studying tractable instances of NP-hard problems. In this paper we improve several results of Freitag et al. [2022] and make progress on the main problem left open there.

In the special case of Euclidean spaces, Freitag et al. [2022] showed, under high enough levels of stability, that no Steiner points occur in the optimal solution. We improve the results there in several ways. First, the assumption about the metric space is relaxed from Euclidean space to a metric space of a certain *doubling dimension*. Second, we improve the bound on stability in the Euclidean setting. Finally, we demonstrate a range of γ in which there can be Steiner points in the optimal solution, but no two Steiner points are adjacent, that is, the optimal solution is *quasi-bipartite*. This quasi-bipartite stability regime motivates our attack on the main problem left open by Freitag et al. [2022] (page 2, first paragraph of Section 3): to give an efficient algorithm for finding the optimal Steiner tree for some value of γ less than 2.

In Section 4 we give an efficient algorithm for instances of the Steiner tree problem whose optimum is quasi-bipartite and which are γ-stable when $\gamma^2(\gamma-1)^2/2 > 1$. With no stability assumption, even the quasi-bipartite variant of the Steiner tree problem is known to be NP-hard [Bern and Plassmann, 1989, Chlebík and Chlebíková, 2002, Gröpl et al., 2002] (indeed, there is a natural reduction from finding a minimum vertex cover of a graph to the bipartite version of the Steiner tree problem). Our main result removes the structural restriction of quasi-bipartiteness of the optimum, but introduces a dependence on L, the maximum degree in the induced graph on the Steiner nodes of the optimal solution. The algorithm requires a stability threshold of $\gamma > \max(\gamma^*, 2-1/L)$, where $\gamma^* \approx 1.8192$ is the positive root of $\gamma(\gamma-1)^3 = 1$. Because this threshold approaches 2 for large L, we leave as an open problem whether there is a uniform threshold $\gamma_0 < 2$ such that there is a polynomial-time algorithm which can find the optimal solution of any γ-stable instance of the metric Steiner tree problem for $\gamma > \gamma_0$.

The stability thresholds established in this paper are summarized in Table 1. The table illustrates the progression from geometric restrictions, such as bounded doubling dimension and Euclidean metrics, to structural assumptions on the optimal Steiner tree, and finally to the general sparse-Steiner regime studied in this paper.

Table 1: Polynomial-time solvable regimes under γ-stability.

Setting	**Sufficient stability condition**
OPT Steiner degree $\leq L$	$\gamma(\gamma-1)^3 > 1, \quad \gamma > 2 - \frac{1}{L}$
Quasi-bipartite	$\gamma^2(\gamma-1)^2/2 > 1$
Euclidean $\mathbb{R}^d$	$\gamma \geq 41/28$
Doubling dimension k	$\frac{2}{2-\gamma} > \left(\frac{4}{\gamma(\gamma-1)}\right)^k$

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2 Preliminaries

Throughout the paper we work with complete finite metric instances. Thus an instance is a finite metric space (V, d) together with a terminal set $T \subseteq V$; feasible Steiner trees are trees in the complete graph on V that span T, with edge weight $d(u, v)$. Distinct vertices are assumed to have positive distance, identifying zero-distance vertices if necessary. Passing from an arbitrary weighted graph to its metric closure preserves the ordinary optimization problem, but not automatically the perturbation model, so stability below is always interpreted for this complete metric instance. When we say Euclidean, we mean the finite version: V is a finite subset of $\mathbb{R}^d$, distances are Euclidean, and Steiner points must lie in $V \setminus T$.

Definition 2.1 (Steiner tree, quasi-bipartiteness) *For a complete finite metric instance (V, T, d), a* Steiner tree *is a tree S on vertices from V that spans all terminals T. Its weight is $w(S) = \sum_{uv \in S} d(u, v)$. We write* OPT *for the minimum-weight Steiner tree:*

$$\mathrm{OPT} \in \arg\min_{S} \sum_{uv \in S} d(u, v).$$

A Steiner tree with no Steiner–Steiner edges is called quasi-bipartite.

Definition 2.2 (Bilu–Linial stability [2012]) *Let $I = (V, T, d)$ be a complete finite metric Steiner tree instance and $\gamma > 1$. A perturbation is a symmetric weight function d' on unordered pairs satisfying*

$$d(u, v) \le d'(u, v) \le \gamma d(u, v) \qquad \forall\, u \ne v.$$

The instance is γ-stable *if, for every such perturbation d', the tree* OPT *is the unique optimum under d'.*

Perturbations are not required to remain metric or Euclidean. As an immediate consequence of the definition, a γ-stable instance must possess a strictly unique optimum OPT. We will also use monotonicity: if an instance is γ-stable, then it is $\hat{\gamma}$-stable for every $1 < \hat{\gamma} \le \gamma$.

We recall the stability facts used throughout.

Lemma 2.3 (Lemma 3 of Freitag et al. [2022]) *Let $1 < \gamma < 2$. In a γ-stable instance, every Steiner point in* OPT *has degree strictly greater than $2/(2 - \gamma)$.*

Lemma 2.4 (Lemma 5 of Freitag et al. [2022]) *Suppose that $ab, bc \in$ OPT. Then:*

a) $d(a, c) > \gamma \max\{d(a, b), d(b, c)\}$;

b) $(2/\gamma)d(a, c) > d(a, b) + d(b, c)$;

c) $(\gamma - 1)d(a, b) < d(b, c)$ and $(\gamma - 1)d(b, c) < d(a, b)$.

Lemma 2.5 *Let $x_0, x_1, \dots, x_k$ be a path in* OPT *with $k \ge 2$. Then*

$$d(x_0, x_k) > \gamma \max_{1 \le i \le k} d(x_{i-1}, x_i).$$

Proof. Fix an edge $e = x_{i-1}x_i$ on the path. Adding x_0x_k to OPT and removing e gives another feasible Steiner tree. If $d(x_0, x_k) \le \gamma d(e)$, then after perturbing only e by a factor of γ, this alternative tree has weight at most the perturbed weight of OPT, contradicting uniqueness under perturbation. Hence $d(x_0, x_k) > \gamma d(e)$ for every edge e on the path. $\square$

Lemma 2.6 *For a γ-stable finite Euclidean instance with $1 < \gamma < 2$, let $xs, sy \in$ OPT be two distinct edges incident to s. Then*

$$\angle xsy > 2 \arcsin(\gamma/2).$$

Proof. Let $r = \|x - s\|$, $R = \|y - s\|$, assume $r \le R$, and put $t = r/R$. By Lemma 2.4(a), $\|x - y\| > \gamma R$. If $\alpha = \angle xsy$, the law of cosines gives

$$\cos\alpha < \frac{t^2 + 1 - \gamma^2}{2t} \le 1 - \frac{\gamma^2}{2},$$

where the second inequality is equivalent to $(t - 1)(t - 1 + \gamma^2) \le 0$. Since $1 - \gamma^2/2 = \cos(2\arcsin(\gamma/2))$, the claim follows. $\square$

3 Steiner trees in Euclidean and doubling spaces

Stability forces the neighbors of any optimal Steiner point to be numerous, length-comparable, and mutually far apart. In doubling metrics this is incompatible with a packing bound; in Euclidean space sharper shortcut and angular arguments give stronger thresholds, as shown in Table 1.

3.1 Steiner trees in doubling metrics

Definition 3.1 (doubling dimension) *A metric space (V, d) has* doubling dimension *at most k if every ball of radius ρ can be covered by at most 2^k balls of radius $\rho/2$.*

The packing bound used below is Lemma A.1 in the appendix.

Theorem 3.2 *Let (V, d) be a metric space of doubling dimension at most k, and let $1 < \gamma < 2$. Consider a γ-stable metric Steiner tree instance with unique optimum* OPT. *If*

$$\frac{2}{2 - \gamma} > \left(\frac{4}{\gamma(\gamma - 1)}\right)^k,$$

then OPT *contains no Steiner points. Consequently,* OPT *is a minimum spanning tree on the terminal set.*

Proof. Assume that OPT contains a Steiner point s, and let $N(s)$ be its neighbors in OPT. By Lemma 2.3, $|N(s)| > 2/(2 - \gamma)$. Let $R = \max_{u \in N(s)} d(s, u)$ and $\rho = \min_{u \in N(s)} d(s, u)$. Lemma 2.4(c) applied to a shortest and a longest edge incident to s gives $\rho > (\gamma - 1)R$.

For distinct $u, v \in N(s)$, Lemma 2.4(a) applied to $u - s - v$ gives

$$d(u,v) > \gamma \max\{d(s,u), d(s,v)\} > \gamma(\gamma-1)R.$$

Thus $N(s) \subseteq B(s,R)$ is δ-separated for $\delta = \gamma(\gamma-1)R$. Since $1 < \gamma < 2$, $\delta < 2R$, so Lemma A.1 yields

$$|N(s)| \leq \left(\frac{4R}{\delta}\right)^k = \left(\frac{4}{\gamma(\gamma-1)}\right)^k,$$

contradicting the hypothesis. Hence OPT has no Steiner points. Then it is a terminal spanning tree, and by optimality it is the terminal MST. □

Corollary 3.3 *For each $k \geq 1$, let $\gamma_k \in (1,2)$ be the unique solution to*

$$\frac{2}{2-\gamma_k} = \left(\frac{4}{\gamma_k(\gamma_k-1)}\right)^k.$$

Then every γ-stable metric Steiner tree instance in a metric of doubling dimension at most k has no Steiner points whenever $\gamma > \gamma_k$.

Proof. The left side is strictly increasing on $(1,2)$, from 2 to $+\infty$, while the right side is strictly decreasing, from $+\infty$ to 2^k; hence the solution is unique. The result is Theorem 3.2. □

For example, $\gamma_1 \approx 1.5616$, $\gamma_2 \approx 1.7688$, and $\gamma_3 \approx 1.8673$. Since Euclidean $\mathbb{R}^d$ has doubling dimension $\Theta(d)$, this gives a dimension-dependent no-Steiner-point criterion for Euclidean instances as well.

3.2 An improved Euclidean bound for when there are no Steiner points

Theorem 3.4 *Let I be a γ-stable finite Euclidean Steiner tree instance in $\mathbb{R}^d$, with unique optimum* OPT. *If $\gamma \geq 41/28 \approx 1.464$, then* OPT *contains no Steiner points. Consequently,* OPT *is the MST on the terminals.*

Proof. By monotonicity, it suffices to prove the theorem for $41/28 \leq \gamma < 3/2$. Suppose that OPT contains a Steiner point s of degree m, with neighbors $p_1, \ldots, p_m$. Set $r_i = \|p_i - s\|$ and $W = \sum_i r_i$. Lemma 2.4(c) gives $\max_i r_i / \min_i r_i \leq 1/(\gamma-1)$. Applying Lemma A.2 to $x_i = p_i - s$ gives a neighbor p_j such that

$$\sum_{i \neq j} \|p_i - p_j\| \leq \frac{\gamma}{\sqrt{2(\gamma-1)}}\sqrt{\frac{m-1}{m}}\, W.$$

Deleting s and its incident edges and reconnecting the resulting components by this star centered at p_j gives another feasible Steiner tree. Under the perturbation multiplying exactly the m edges incident to s by γ, stability implies that this replacement star has weight strictly larger than γW. Hence

$$\frac{\gamma}{\sqrt{2(\gamma-1)}}\sqrt{\frac{m-1}{m}} > \gamma,$$

which, since $\gamma < 3/2$, implies $m > 1/(3-2\gamma)$. Thus $\gamma \geq 41/28$ gives $m \geq 15$.

On the other hand, Lemma 2.6 and Lemma A.3 applied to the unit vectors $(p_i - s)/\|p_i - s\|$ give

$$m < \frac{\gamma^2}{\gamma^2 - 2} \leq \frac{(41/28)^2}{(41/28)^2 - 2} = \frac{1681}{113} < 15,$$

a contradiction. Hence OPT has no Steiner points and is the terminal MST. □

3.3 Planar Euclidean quasi-bipartiteness

For a path in OPT, Lemma 2.5 gives

$$\|x_0 - x_k\| > \gamma \max_{1 \leq i \leq k} \|x_i - x_{i-1}\|. \tag{1}$$

For two distinct neighbors of a Steiner vertex, Lemma 2.6 gives separation by

$$\theta_\gamma := 2\arcsin(\gamma/2). \tag{2}$$

Let γ_{QB} be the unique root in $(4/3, 11/8)$ of

$$1 - 2\cos(6\arcsin(\gamma/2)) = \gamma,$$

equivalently of $\gamma^6 - 6\gamma^4 + 9\gamma^2 - \gamma - 1 = 0$. Numerically, $\gamma_{\mathrm{QB}} \approx 1.370$.

Theorem 3.5 *Let I be a γ-stable finite Euclidean Steiner tree instance in $\mathbb{R}^2$, with unique optimum* OPT. *If $\gamma \geq \gamma_{\mathrm{QB}}$, then* OPT *is quasi-bipartite.*

Proof. By monotonicity, it suffices to consider $\gamma_{\mathrm{QB}} \leq \gamma < 11/8$. Suppose that OPT contains a Steiner–Steiner edge st, scaled and rotated so that $s = (0,0)$ and $t = (1,0)$. Put $\theta = 2\arcsin(\gamma/2)$. Since $\gamma > 4/3$, Lemma 2.3 implies that every Steiner vertex has degree at least 4.

At s, choose the non-t neighbor u whose ray from s has smallest positive angle ϕ from the positive x-axis. Since there are at least three non-t directions and they are pairwise separated by more than θ, while each is separated from the direction st by more than θ, we have $\theta < \phi < 2\pi - 3\theta$. Write $u = (r\cos\phi, r\sin\phi)$. Similarly, choose the non-s neighbor v of t whose ray from t has largest angle ψ relative to the positive x-axis. Then $3\theta - \pi < \psi < \pi - \theta$, and $v = (1 + q\cos\psi, q\sin\psi)$.

Applying (1) to the paths $u - s - t$ and $s - t - v$ gives

$$\|u - t\| > \gamma \max\{r, 1\}, \qquad \|v - s\| > \gamma \max\{q, 1\}.$$

Hence Lemma A.4 gives $\|u - v\| \leq \gamma \max\{r, q, 1\}$. But applying (1) to the path $u - s - t - v$ gives the strict reverse inequality, a contradiction. □

4 Polynomial-Time Algorithm for Stable Instances with Sparse Steiner Components

We now give a polynomial-time algorithm for metric instances whose optimum is quasi-bipartite under the stability regime summarized in Table 1. We first handle the quasi-bipartite case, and then extend the argument to the more general sparse-Steiner setting where the maximum Steiner degree in OPT is bounded.

For a subgraph $H \subseteq \mathrm{OPT}$, define its *terminal components* to be the connected components of H that contain a terminal, together with each terminal not present in H as a singleton component. The algorithm below maintains the invariant that every connected component of H contains a terminal. Let $F(a, \bar{b})$ denote a star centered at a with leaves $\bar{b} = \{b_1, \ldots, b_m\}$.

Definition 4.1 (terminal component fan, density) *A* terminal component *fan relative to H is a star $F(a, \bar{b})$ such that:*

1. *a is a Steiner vertex not contained in H;*
2. *the leaves b_i lie in distinct terminal components of H;*
3. *$m \geq 2$;*
4. *the edge weights in the fan are within a factor $1/(\gamma - 1)$ of each other.*

Its density *is*

$$\mathrm{dens}(F(a, \bar{b})) = \frac{\sum_{i=1}^{m} d(a, b_i)}{m - 1}.$$

The denominator is $m-1$ because such a fan decreases the number of terminal components by $m - 1$.

Lemma 4.2 *Let $H \subseteq \mathrm{OPT}$, and let $F = F(a, \bar{b})$ be a terminal component fan relative to H with m leaves. If $F \nsubseteq \mathrm{OPT}$, then there is a set $R \subseteq \mathrm{OPT} \setminus (H \cup F)$ such that replacing the appropriate edges of OPT by the edges of F gives a feasible Steiner tree, and either:*

1. *$a \in \mathrm{OPT}$, in which case $|R| = |F \setminus \mathrm{OPT}|$; or*
2. *$a \notin \mathrm{OPT}$, in which case $|R| = m - 1$.*

Proof. See Appendix B. □

Lemma 4.3 *Assume $H \subseteq \mathrm{OPT}$ and OPT is quasi-bipartite. Let x be a Steiner vertex of OPT not contained in H. Then the set of all edges of OPT incident to x is a terminal component fan relative to H. If this fan has n leaves and density Δ, then every one of its edges has weight greater than*

$$(\gamma - 1)\frac{n-1}{n}\,\Delta.$$

Moreover, $n > 2/(2 - \gamma)$.

Proof. Since OPT is quasi-bipartite, every neighbor of x in OPT is a terminal. No two such terminals can lie in the same terminal component of H, or the two edges from x together with the path in H between the terminals would create a cycle in OPT. Thus the incident edges form a terminal component fan, and the ratio condition follows from Lemma 2.4(c).

If the fan weights are $w_1, \ldots, w_n$ and $S = \sum_i w_i$, then the smallest edge is greater than $(\gamma - 1)S/n = (\gamma - 1)(n - 1)\Delta/n$. Finally, $n = \deg_{\mathrm{OPT}}(x) > 2/(2 - \gamma)$ by Lemma 2.3. □

Lemma 4.4 *Let $1 < \gamma < 2$ and $\gamma^2(\gamma - 1)^2/2 > 1$, and suppose that $H \subseteq \mathrm{OPT}$ and OPT is quasi-bipartite. Suppose that $F = F(a, \bar{b})$ is a terminal component fan relative to H such that*

- *$\mathrm{dens}(F)$ is at most the weight of every edge not in H connecting two terminal components of H, and*
- *$\mathrm{dens}(F)$ is minimal among all terminal component fans relative to H.*

Then $F \subseteq \mathrm{OPT}$.

Proof. Let $\delta = \mathrm{dens}(F)$, let m be the number of leaves of F, and suppose $F \nsubseteq \mathrm{OPT}$. Every edge of F has weight less than $\delta/(\gamma - 1)$: if M is the largest fan-edge weight, then all other fan edges have weight at least $(\gamma - 1)M$, so $w(F) \geq M + (m-1)(\gamma-1)M > (m-1)(\gamma-1)M$. Since $w(F) = (m - 1)\delta$, we get $M < \delta/(\gamma - 1)$.

Let $R \subseteq \mathrm{OPT} \setminus (H \cup F)$ be from Lemma 4.2. For $e \in R$, if e connects two terminal components of H, then $d(e) \geq \delta$, so $\gamma d(e) > \delta/(\gamma - 1)$ because $\gamma(\gamma - 1) > 1$. Otherwise e is incident to a Steiner vertex $x \notin H$. By Lemma 4.3, the incident fan at x has some density $\Delta \geq \delta$ and $n > 2/(2 - \gamma)$ leaves, so

$$d(e) > (\gamma - 1)\frac{n-1}{n}\Delta > (\gamma - 1)\frac{\gamma}{2}\delta.$$

After perturbing e by γ, its weight is greater than $\gamma^2(\gamma - 1)\delta/2 > \delta/(\gamma - 1)$.

Thus, under the perturbation multiplying every edge of R by γ, every edge in R is heavier than every edge of F. If $a \in \mathrm{OPT}$, then $|R| = |F \setminus \mathrm{OPT}|$, and replacing R by the missing fan edges gives a strictly cheaper tree. If $a \notin \mathrm{OPT}$, then $|R| = m - 1$, and every perturbed edge of R has weight greater than δ, so the perturbed weight of R is greater than $(m - 1)\delta = w(F)$. Both cases contradict stability. □

Lemma 4.5 *Let $1 < \gamma < 2$ and $\gamma^2(\gamma - 1)^2/2 > 1$, and suppose that $H \subseteq \mathrm{OPT}$ and OPT is quasi-bipartite. Let e be an edge connecting two terminal components of H such that*

- *e has minimum weight among such edges, and*
- *$d(e)$ is at most the density of every terminal component fan relative to H.*

Then $e \in \mathrm{OPT}$.

Proof. Assume $e \notin \mathrm{OPT}$. Adding e to OPT creates a cycle containing an edge $e' \in \mathrm{OPT} \setminus H$ whose removal leaves a feasible Steiner tree. If e' connects two terminal components, then $d(e') \geq d(e)$, and perturbing e' by γ makes replacement by e strictly cheaper. Otherwise e' is incident to a Steiner vertex $x \notin H$. By Lemma 4.3, the incident fan at x has density $\Delta \geq d(e)$ and $n > 2/(2-\gamma)$ leaves, so

$$d(e') > (\gamma-1)\frac{n-1}{n}\Delta > (\gamma-1)\frac{\gamma}{2}d(e).$$

After perturbing e' by γ, its weight is greater than $\gamma^2(\gamma-1)d(e)/2 > d(e)$, because $\gamma^2(\gamma-1)^2/2 > 1$. This contradicts stability. □

Theorem 4.6 *Given $\gamma \in (1,2)$ with*

$$\gamma^2(\gamma-1)^2/2 > 1,$$

every γ-stable instance whose optimum is quasi-bipartite can be solved in polynomial time.

Proof. Initialize $H = \emptyset$. At each step, compute a minimum-density terminal component fan relative to H, if one exists, and a minimum-weight edge connecting two terminal components. Add to H whichever attains the smaller value, breaking ties arbitrarily, and repeat until all terminals lie in one terminal component.

The invariant $H \subseteq \mathrm{OPT}$ holds by induction. If the chosen structure is a fan, Lemma 4.4 applies; if it is an edge, Lemma 4.5 applies. Hence each iteration decreases the number of terminal components, so there are at most $|T|-1$ iterations. At termination, $H \subseteq \mathrm{OPT}$ spans all terminals. Since all edge weights are positive and OPT is optimal, no proper subgraph of OPT can span all terminals; otherwise deleting the positive-weight removed edges would give a cheaper feasible tree. Thus $H = \mathrm{OPT}$.

Both minimum structures are found in polynomial time. Let $n = |V|$, $m_e = |E|$, and $|T|$ be the total number of terminal vertices. At a given iteration, let $k \leq |T|$ be the number of distinct terminal components. The minimum-weight component-connecting edge is found in $O(m_e)$ time by scanning all edges. To find the optimal fan, we evaluate $O(n)$ candidate centers $a \in V \setminus T$ not contained in H. For a fixed center a, there are at most $|T|$ edges connecting a to a terminal component; we pick each to establish the assumed minimum weight w of the fan. For a fixed w, the allowed interval is $[w, w/(\gamma-1)]$. In $O(|T|)$ time, we scan all edges from a to the terminal components that fall into this interval, keeping only the single cheapest edge for each distinct terminal component. Sorting this resulting list of at most k edges by weight takes $O(k \log k)$ time. Then, calculating the density for the fans corresponding to each valid size $m \geq 2$ by taking the first m edges in the sorted list takes $O(k)$ time. Evaluating a single center over all its starting weights w therefore takes $O(|T|(|T| + k\log k))$ time. Picking the fan with the smallest density across all $O(n)$ centers takes $O(n|T|^2 \log|T|)$ time per iteration. Over at most $|T|$ iterations, the total time for computing fans is bounded by $O(n|T|^3 \log|T|)$, yielding an overall algorithm complexity safely within $O(n^4 \log n)$. □

We now extend the algorithm beyond the quasi-bipartite setting to the sparse-Steiner regime in Table 1. Specifically, we assume that each Steiner vertex has at most L Steiner neighbors in OPT.

Lemma 4.7 *Suppose $H \subseteq \mathrm{OPT}$, and in OPT every Steiner vertex is adjacent to at most $L \geq 1$ other Steiner vertices. Let $1 < \gamma < 2$ and $\gamma > 2 - 1/L$. If x is a Steiner vertex of OPT not contained in H, then x has a terminal component fan F_x relative to H with $n > 1/(2-\gamma)$ leaves. If $\Delta = \mathrm{dens}(F_x)$, then every edge of OPT incident to x has weight greater than*

$$(\gamma-1)\frac{n-1}{n}\Delta > (\gamma-1)^2\Delta.$$

Proof. Lemma 2.3 gives $\deg_{\mathrm{OPT}}(x) > 2/(2-\gamma)$. At most $L < 1/(2-\gamma)$ neighbors are Steiner vertices, so x has more than $1/(2-\gamma)$ terminal neighbors. As before, these terminals lie in distinct terminal components of H, and their incident edges form a fan.

For any edge e incident to x, Lemma 2.4(c) gives $d(e) > (\gamma-1)d(f)$ for every edge f in F_x. Averaging over the fan edges gives $d(e) > (\gamma-1)(n-1)\Delta/n$. Since $n > 1/(2-\gamma)$, we have $(n-1)/n > \gamma - 1$. □

Lemma 4.8 *Suppose $H \subseteq \mathrm{OPT}$, and in OPT every Steiner vertex is adjacent to at most $L \geq 1$ other Steiner vertices. Let $1 < \gamma < 2$, $\gamma(\gamma-1)^3 > 1$, and $\gamma > 2 - 1/L$. Suppose that $F = F(a, \bar{b})$ is a terminal component fan relative to H such that*

- $\mathrm{dens}(F)$ *is at most the weight of every edge not in H connecting two terminal components of H, and*
- $\mathrm{dens}(F)$ *is minimal among all terminal component fans relative to H.*

Then $F \subseteq \mathrm{OPT}$.

Proof. Let $\delta = \mathrm{dens}(F)$ and suppose $F \nsubseteq \mathrm{OPT}$. As in the proof of Lemma 4.4, every edge of F has weight less than $\delta/(\gamma-1)$. Let R be from Lemma 4.2. For $e \in R$, if e connects two terminal components, then $d(e) \geq \delta$, so $\gamma d(e) > \delta/(\gamma-1)$, since $\gamma(\gamma-1) > 1$ follows from $\gamma(\gamma-1)^3 > 1$. Otherwise e is incident to a Steiner vertex $x \notin H$, and Lemma 4.7 gives a fan at x of density $\Delta \geq \delta$ with $d(e) > (\gamma-1)^2\Delta \geq (\gamma-1)^2\delta$. After perturbation, $\gamma d(e) > \gamma(\gamma-1)^2\delta > \delta/(\gamma-1)$.

Thus every perturbed edge of R is heavier than every edge of F. The same cardinality and total-weight comparisons as in Lemma 4.4 contradict stability. □

Lemma 4.9 *Suppose $H \subseteq \mathrm{OPT}$, and in OPT every Steiner vertex is adjacent to at most $L \geq 1$ other Steiner* *vertices. Let $1 < \gamma < 2$, $\gamma(\gamma-1)^3 > 1$, and $\gamma > 2-1/L$. Let e be an edge connecting two terminal components of H such that*

- *e has minimum weight among such edges, and*
- *$d(e)$ is at most the density of every terminal component fan relative to H.*

Then $e \in \mathrm{OPT}$.

Proof. Assume $e \notin \mathrm{OPT}$, and let $e' \in \mathrm{OPT} \setminus H$ be an edge on the cycle created by adding e whose removal keeps a feasible tree. If e' connects two terminal components, then $d(e') \geq d(e)$ and perturbing e' makes replacement by e cheaper. Otherwise e' is incident to a Steiner vertex $x \notin H$. By Lemma 4.7, the fan at x has density $\Delta \geq d(e)$ and $d(e') > (\gamma-1)^2\Delta$. After perturbing e' by γ, its weight is greater than $\gamma(\gamma-1)^2 d(e) > d(e)$, because $\gamma(\gamma-1)^3 > 1$. This contradicts stability. □

Corollary 4.10 *Let $L \geq 1$, and suppose that in* OPT *every Steiner vertex is adjacent to at most L other Steiner vertices. Given $\gamma \in (1, 2)$ such that the instance is γ-stable,*

$$\gamma(\gamma-1)^3 > 1 \quad \text{and} \quad \gamma > 2 - 1/L,$$

the instance can be solved in polynomial time.

Proof. The algorithm in Theorem 4.6 applies without modification, with Lemmas 4.8 and 4.9 replacing Lemmas 4.4 and 4.5, respectively. The running time remains bounded by $O(n^4 \log(n))$. □

The execution of the algorithm does not require prior knowledge of the maximum Steiner degree L, nor does its running time depend on L. At each step, the search for minimum-density fans and component-connecting edges relies only on the stability parameter γ to define the valid edge-weight intervals. The only role of the parameter L is to define the degree-dependent condition $\gamma > 2 - 1/L$, which, together with the other condition $\gamma(\gamma-1)^3 > 1$, guarantees that the algorithm returns the true optimum. Since L is trivially bounded by the total number of Steiner nodes, the polynomial runtime holds in the worst case as well, with the caveat that the required stability threshold approaches 2 as the number of Steiner nodes increases.

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A Technical lemmas

Lemma A.1 *Let (V, d) have doubling dimension at most k. Let $X \subseteq B(x, R)$ and assume $0 < \delta \leq 2R$ and $d(u, v) > \delta$ for all distinct $u, v \in X$. Then*

$$|X| \leq \left(\frac{4R}{\delta}\right)^k.$$

Proof. Choose $m = \lceil \log_2(2R/\delta) \rceil$. By iterating the doubling property, $B(x, R)$ is covered by at most 2^{km} balls of radius $R/2^m \leq \delta/2$. Each such ball contains at most one point of X, since two points in it would be at distance at most δ. Hence

$$|X| \leq 2^{km} \leq 2^{k(\log_2(2R/\delta)+1)} = \left(\frac{4R}{\delta}\right)^k.$$

□

Lemma A.2 *Let $x_1, \ldots, x_m \in \mathbb{R}^d$, let $r_i = \|x_i\|$, and let $W = \sum_i r_i$. If $\max_i r_i / \min_i r_i \leq \kappa$, where $\kappa \geq 1$, then there is an index j such that*

$$\sum_{i \neq j} \|x_i - x_j\| \leq \frac{\kappa+1}{\sqrt{2\kappa}} \sqrt{\frac{m-1}{m}}\, W.$$

Proof. We have

$$\sum_{i<j} \|x_i - x_j\|^2 = m \sum_i \|x_i\|^2 - \left\| \sum_i x_i \right\|^2 \leq m \sum_i r_i^2.$$

By Cauchy–Schwarz,

$$\frac{2}{m}\sum_{i<j}\|x_i - x_j\| \le \sqrt{2(m-1)\sum_i r_i^2}.$$

The left side is the average over j of $\sum_{i\neq j}\|x_i - x_j\|$, so some j satisfies this bound. Finally, since the r_i lie in an interval of ratio at most κ, the Kantorovich second-moment inequality [Greub and Rheinboldt, 1959] gives

$$\frac{\sum_i r_i^2}{(\sum_i r_i)^2} \le \frac{(\kappa+1)^2}{4\kappa m},$$

and substitution proves the lemma. □

Lemma A.3 *Let $u_1, \dots, u_m$ be unit vectors in a Euclidean space. Suppose that $u_i \cdot u_j < 1 - \gamma^2/2$ for every $i \neq j$, where $\gamma > \sqrt{2}$. Then*

$$m < \frac{\gamma^2}{\gamma^2 - 2}.$$

Proof. Since $1 - \gamma^2/2 < 0$,

$$0 \le \left\|\sum_i u_i\right\|^2 = m + 2\sum_{i<j} u_i \cdot u_j < m + m(m-1)\left(1 - \frac{\gamma^2}{2}\right).$$

Rearranging gives the claim. □

Lemma A.4 *Let $\gamma_{\mathrm{QB}} \le \gamma < 11/8$ where $\gamma_{\mathrm{QB}} \approx 1.370$ is the unique root in $(4/3, 11/8)$ of*

$$1 - 2\cos(6\arcsin(\gamma/2)) = \gamma,$$

and let $\theta = 2\arcsin(\gamma/2)$. Let

$$u = (r\cos\phi, r\sin\phi), \qquad v = (1 + q\cos\psi, q\sin\psi),$$

where $r, q > 0$, and suppose $\theta < \phi < 2\pi - 3\theta$ and $3\theta - \pi < \psi < \pi - \theta$. If

$$\|u - (1,0)\| > \gamma\max\{r,1\}, \qquad \|v\| > \gamma\max\{q,1\},$$

then

$$\|u - v\| \le \gamma\max\{r,q,1\}.$$

Proof. Let $D = \|u - v\|$. Expanding the squared distance directly yields

$$D^2 = 1 + r^2 + q^2 + 2q\cos\psi - 2r\cos\phi - 2rq\cos(\phi - \psi).$$

First, we bound the trigonometric terms. Define $c = \cos(3\theta - \pi) = -\cos(3\theta)$. The assumption on γ restricts all the angles within bounds where the cosine function is monotone. Thus we have

$$\cos\psi \le \cos(3\theta - \pi) = c, \quad -\cos\phi \le -\cos(2\pi - 3\theta) = c,$$

and

$$-\cos(\phi - \psi) \le -\cos(3\pi - 6\theta) = -(1 - 2c^2).$$

Applying these bounds gives:

$$D^2 \le F(r,q) := (1 + c(r+q))^2 + (1 - c^2)(r - q)^2. \tag{3}$$

Next, we derive the bounds for r and q. The distance assumption $\|u - (1,0)\| > \gamma\max\{r,1\}$ is equivalent to $\sqrt{r^2 + 1 - 2r\cos\phi} > \gamma\max\{r,1\}$. Solving this separately for $r \le 1$ and $r \ge 1$ yields the condition $a(\cos\phi) < r < a(\cos\phi)^{-1}$, where $a(x) = x + \sqrt{x^2 + \gamma^2 - 1}$. Since $\cos\phi > -c$ and $a(x)$ is an increasing function, we introduce $\rho = a(-c) = \sqrt{c^2 + \gamma^2 - 1} - c$. Thus, $\rho < r < \rho^{-1}$. The same reasoning applies to v, giving $\rho < q < \rho^{-1}$. Therefore,

$$\rho < r, q < \rho^{-1}. \tag{4}$$

Let $M = \max\{r,q,1\}$. We will show that $F(r,q) \le (1+2c)^2 M^2$ for each case of M, and $1 + 2c < \gamma$. Because $F(r,q)$ is separately convex, its maximum on a bounded domain occurs at the extreme points. If $M = 1$, then $(r,q) \in [\rho,1]^2$. The maximum occurs at a corner. The highest non-trivial corner is $(1,\rho)$, which requires $F(1,\rho) - (1+2c)^2 \le 0$. Expanding this gives the required condition:

$$(\rho - 1)(\rho + 4c^2 + 2c - 1) \le 0. \tag{5}$$

If $M = r$ (the case $M = q$ is symmetric), write $a = 1/r \le 1$ and $b = q/r \in [\rho^2, 1]$. Then

$$\frac{F(r,q)}{r^2} = (a + c(1+b))^2 + (1 - c^2)(1 - b)^2.$$

This expression is strictly increasing in a, so it is maximized at $a = 1$. The remaining function is convex in b, taking its maximum at either $b = 1$ (yielding $(1+2c)^2$) or at the lower boundary $b = \rho^2$. The condition at the lower boundary requires the value not to exceed $(1+2c)^2$, yielding:

$$(\rho^2 - 1)(\rho^2 + 4c^2 + 2c - 1) \le 0. \tag{6}$$

Finally, we determine the values of γ that satisfy these requirements. To satisfy (5), (6), and the final bound $(1+2c)M \le \gamma M$, we need:

$$0 < \rho < 1, \quad 1 + 2c \le \gamma, \quad \rho^2 + 4c^2 + 2c \ge 1.$$

Since $\theta = 2\arcsin(\gamma/2)$, we have $c = \frac{\gamma^6}{2} - 3\gamma^4 + \frac{9\gamma^2}{2} - 1$. On the interval $[\gamma_{\mathrm{QB}}, 11/8]$, $c \in (0,1)$ and its derivative is strictly negative. The constant γ_{QB} is defined where $1 + 2c(\gamma_{\mathrm{QB}}) = \gamma_{\mathrm{QB}}$, so $1 + 2c \le \gamma$ holds. Furthermore, $\gamma > 1$ implies $\rho > 0$, and $\gamma < \sqrt{2}$ implies $\rho < 1$, rendering the $(\rho - 1)$ and $(\rho^2 - 1)$ terms negative. Finally, defining $h(\gamma) = \rho^2 + 4c^2 + 2c - 1$, we find $h'(\gamma) < 0$ on the interval, with $h(11/8) > 0$. Thus, $h(\gamma) \ge 0$ on $[\gamma_{\mathrm{QB}}, 11/8]$, rendering the second factor in our boundary conditions positive. All inequalities hold, proving $D \le \gamma M$. □

B Deferred exchange proof

Proof. [Proof of Lemma 4.2] Contract every terminal component of H to a single vertex. Since $H \subseteq \mathrm{OPT}$ and OPT is a tree, the image of OPT after contraction is a tree on the contracted terminal-component vertices together with the vertices of $\mathrm{OPT} \setminus H$.

If $a \in \mathrm{OPT}$, add the edges of $F \setminus \mathrm{OPT}$ one at a time. Each added edge creates a cycle. Since F is a forest, that cycle contains an edge of the current tree outside F; since the leaves of F lie in distinct contracted terminal components, this edge is outside H. Removing it preserves all fan edges already present and restores a tree. Repeating gives $R \subseteq \mathrm{OPT} \setminus (H \cup F)$ with $|R| = |F \setminus \mathrm{OPT}|$.

If $a \notin \mathrm{OPT}$, adding the m fan edges introduces one new vertex and m new edges. The first attaches the new vertex; each remaining edge creates a cycle containing an old edge outside F and outside H. Removing one such edge for each of these $m - 1$ fan edges gives $R \subseteq \mathrm{OPT} \setminus (H \cup F)$ with $|R| = m - 1$. In both cases the resulting graph is connected on all terminals and acyclic after the specified deletions, so it is a feasible Steiner tree. □