Go is PSPACE hard

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Clique Talk

“Go is Polynomial-Space Hard” David Lichtenstein and Michael Sipser.

Some Complexity Reminders

- PSPACE is the class of languages that are decidable in polynomial space on a deterministic turning machine
- NL $\subseteq$ P $\subseteq$ NP $\subseteq$ PSPACE = NPSPACE $\subseteq$ EXPTIME
- P $\subset$ EXPTIME
- NL $\subset$ PSPACE

TQBF

- Boolean formulas with quantifiers are *quantified Boolean formulas*
  - $\phi_1 = \forall x \exists y [(x \vee y) \wedge (-x \vee -y)]$ is true
  - $\phi_2 = \exists x \forall y [(x \vee y) \wedge (-x \vee -y)]$ is false
- When each variable falls within the scope of some quantifier, the formula is *fully quantified*
- The TQBF problem is to determine whether a fully quantified Boolean formula is true of false
- **Theorem**: *TQBF is PSPACE complete*

Plan of Action

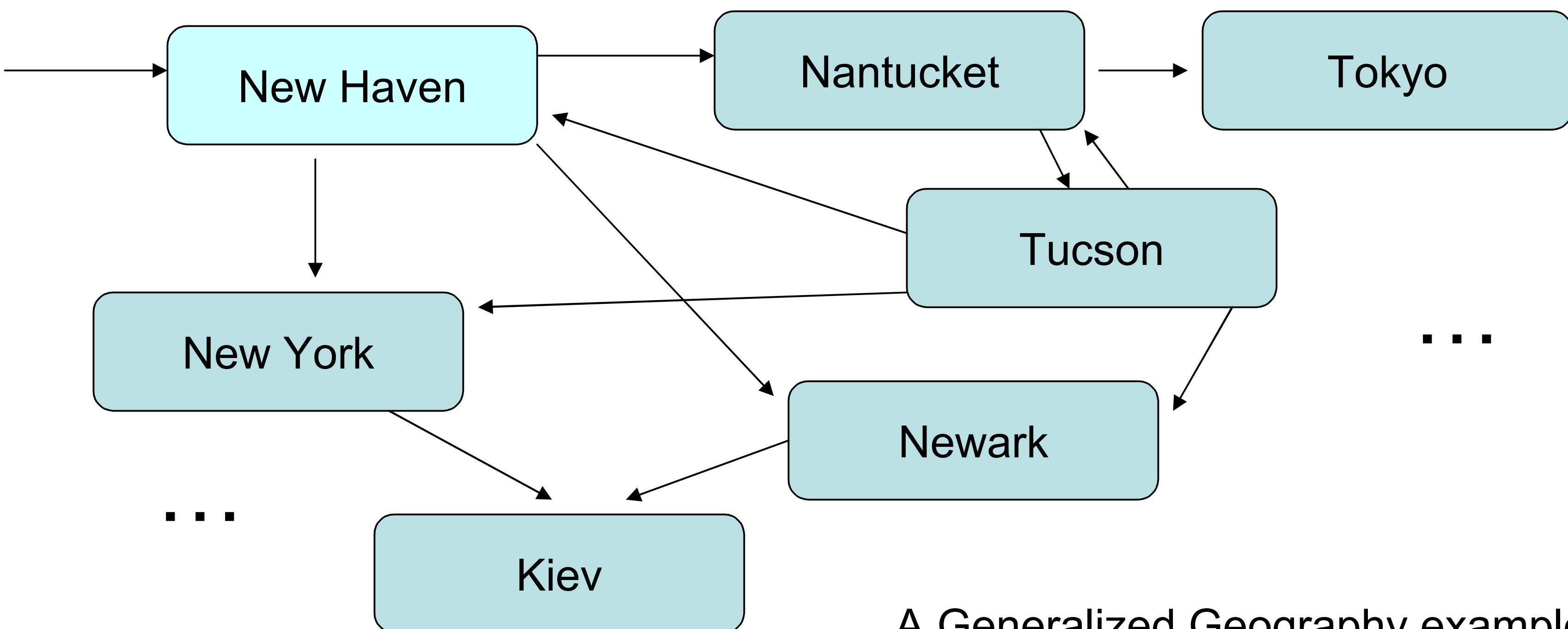
- Reduce TQBF to Generalized Geography
- Reduce Generalized Geography to Planar Generalized Geography
- Teach you the Rules of Go
- Reduce Planar Geography to Go

Generalized Geography (GG)

- 2 player game in which players take turns naming cities from anywhere in the world.
- Each city chosen must begin with the same letter that ended with the previous city’s name.
- Repetition isn’t permitted.
- Game starts with some designated city and ends when a player cannot continue.

GG Continued

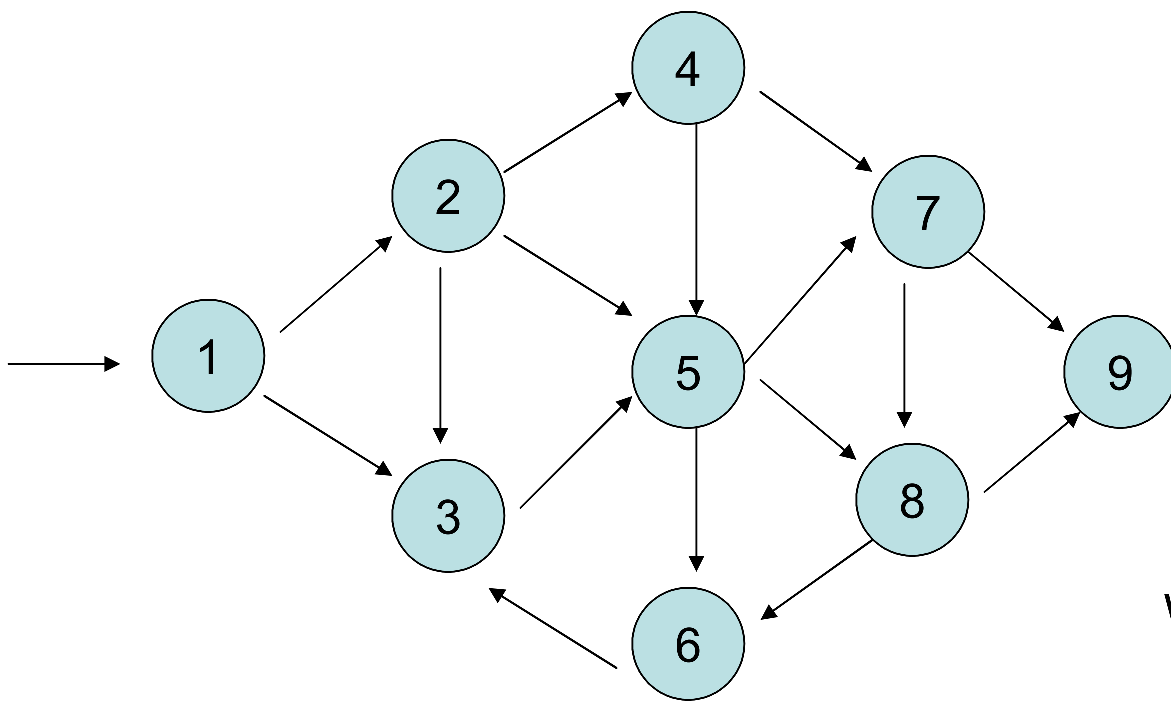
- **Theorem**: *The problem of determining which player has a winning strategy in a generalized geography game is PSPACE-complete.*



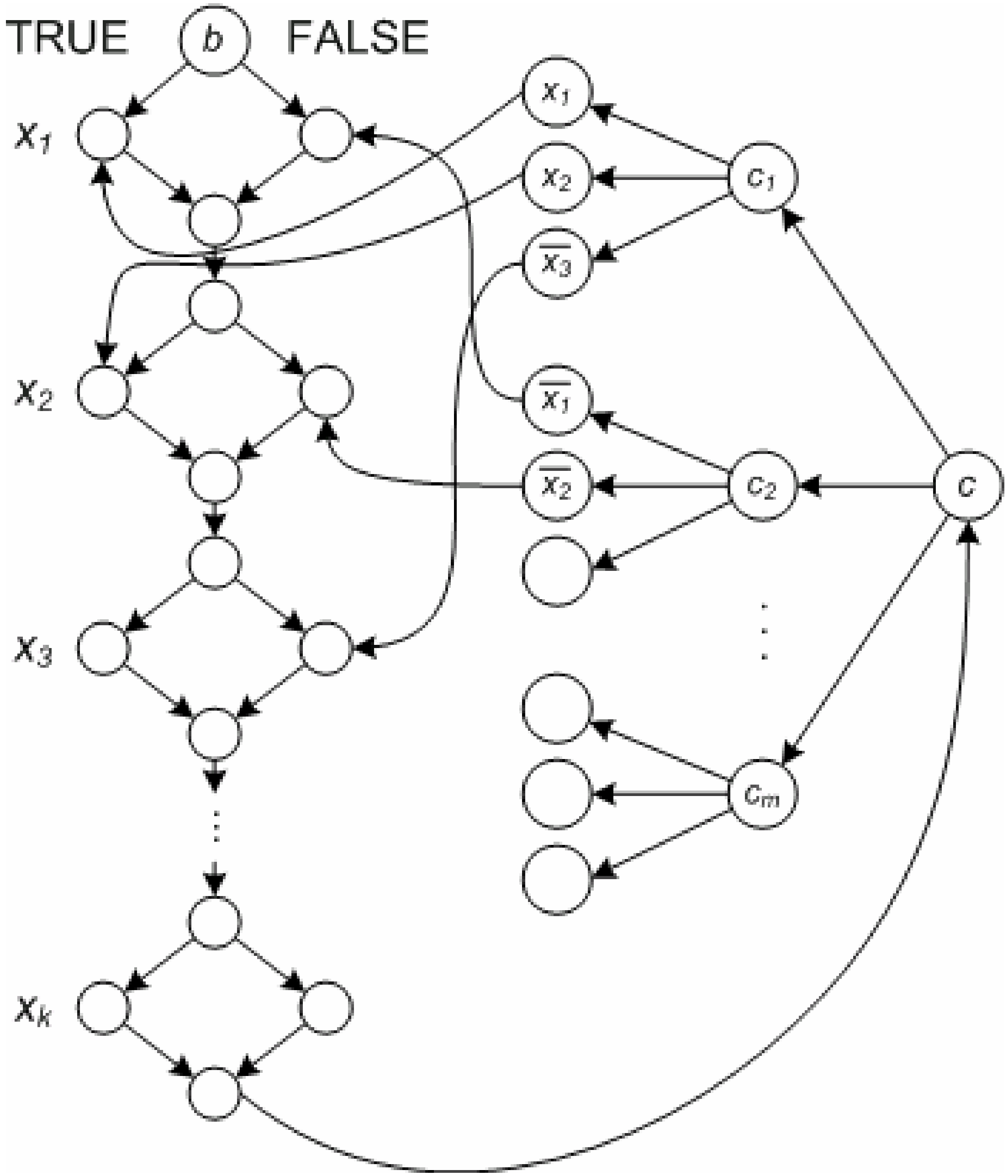
A Generalized Geography example

A Sample Game

- Any directed graph with a designated start node is an example of generalized geography



Who wins?



GG is PSPACE hard

A FQBF can be expressed in the form:
$\phi = \exists x1 \ \forall x2 \ \exists x3 \ \ldots [\ \psi]$
where ψ is in CNF

Player 1 is E, player 2 is A
If ϕ is true, player E wins.

Play starts at b w/ E
At c, it's A's turn

If ϕ is false, player A wins by choosing unsat clause. Else, player E can win by selecting a sat variable.

QED.

Progress Check

- Reduce TQBF to Generalized Geography
- Reduce Generalized Geography to Planar Generalized Geography
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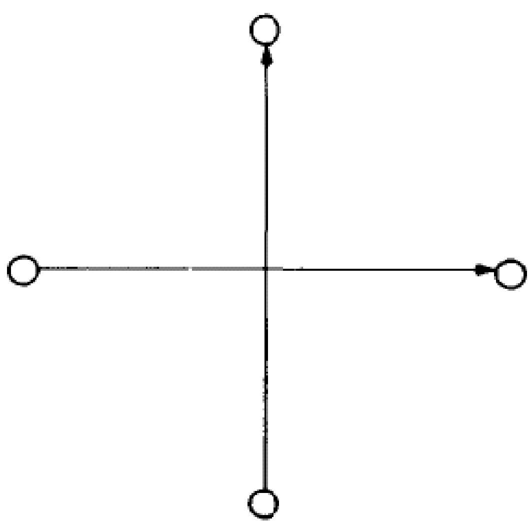
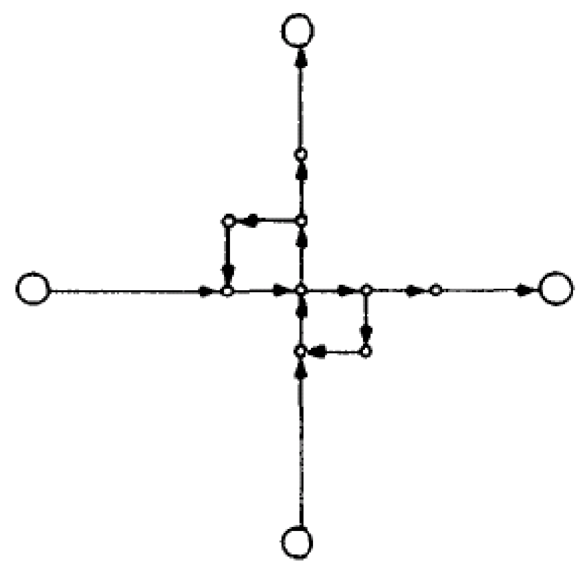
Planar GG is PSPACE hard

- Planar Generalized Geography is GG played on planar graphs
- Draw GG in the plane, allow arcs to cross
- Just replace and QED:



with



Simple case analysis shows this works

Progress Check

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The Rules of Go (from [LS ’80])

- Go is played on a grid of n x n locations called points
- There are 2 players: black and white. Black moves first.
- A player moves by placing a stone of his color on a vacant point or passes.
- The games terminates when both players pass consecutively

More Rules

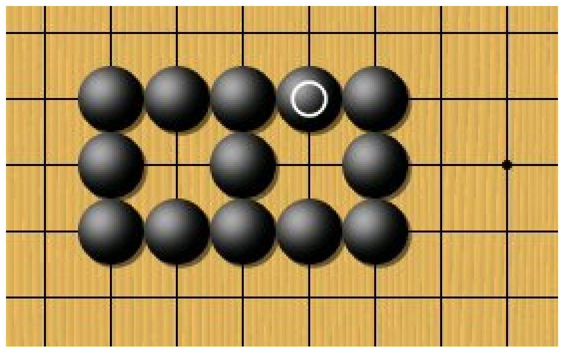
- As the game progresses, stones form clusters called groups
- A group is a maximal contiguous region of uniformly colored stones
- A group of stones becomes surrounded if none of them is adjacent to a vacant stone
- After each black move, all surrounded white stones are removed, followed by all surrounded black stones. (and vice versa)

Scoring

- At the end of the game all dead stones are removed from the board
  - A stone is dead if it ultimately can be surrounded despite any attempts to save it
- Then, a vacant point is white territory if it is surrounded on all sides by either white stones or an edge of the board. (or black)
- The final score for white is the count of white territory plus the number of black stones removed from the board at any time. (and vice versa)
- The player with the highest score wins.

Eyes

- “Two Eyes you’re alive”



- Frequently, in a game, a player may have a nearly surrounded group of stones which he is desperately trying to connect to a group with two eyes. His opponent is trying to cut him off. The proof exploits this situation.

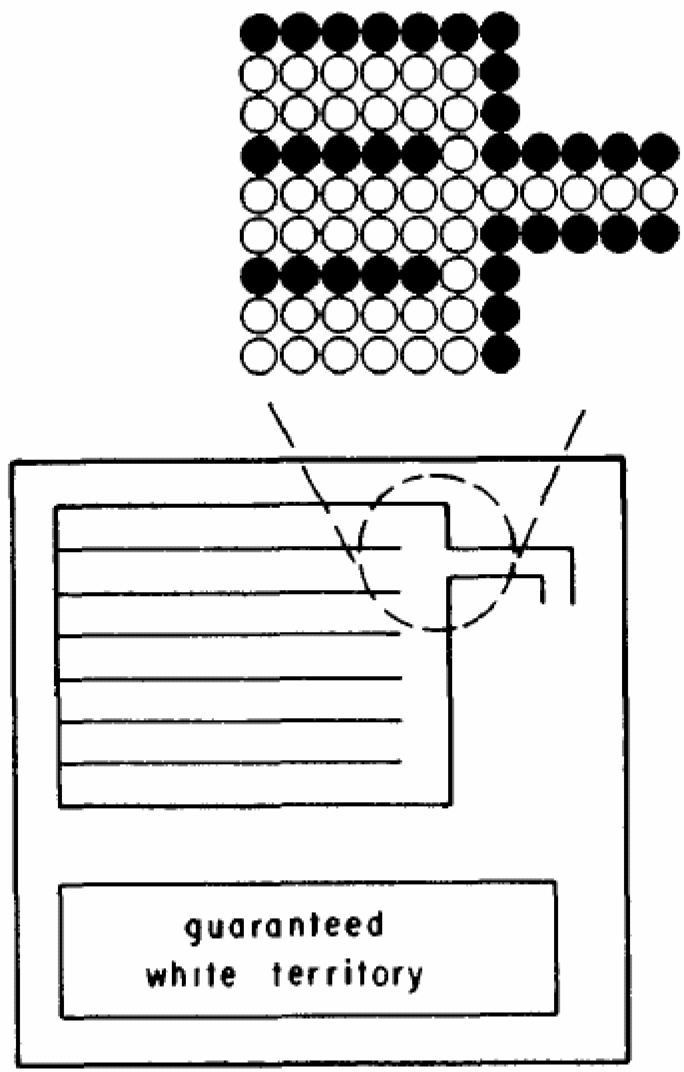
Progress Check

- Reduce TQBF to Generalized Geography
- Reduce Generalized Geography to Planar Generalized Geography
- Teach you the Rules of Go
- Reduce Planar Geography to Go

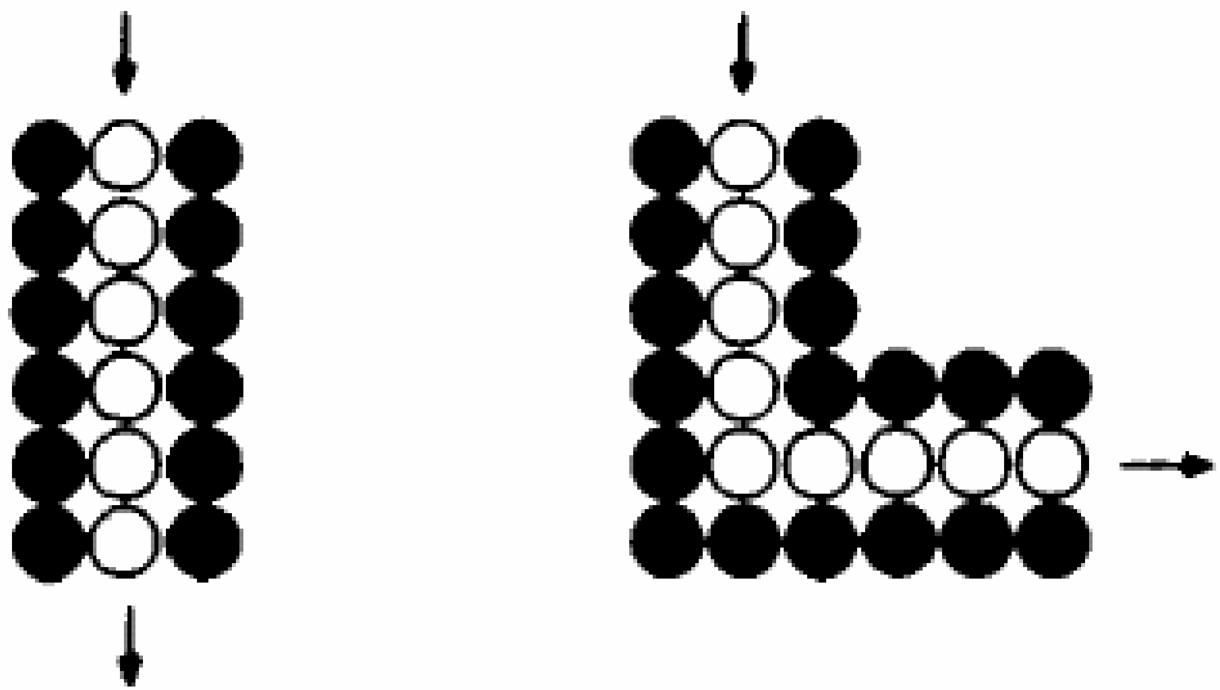
The Reduction

- We will make a Go position that will have the property that Black has a winning strategy only if Planar GG player E does.
- Proof will make use of the following ideas:
  - white will have a lot of territory
  - white will have an even larger group trying to escape capture through a “breach” in blacks wall
  - outcome of the game will hinge on whether black can take the white group
  - the breach leads to a planar GG structure
  - B and W will be forced to play a geography game

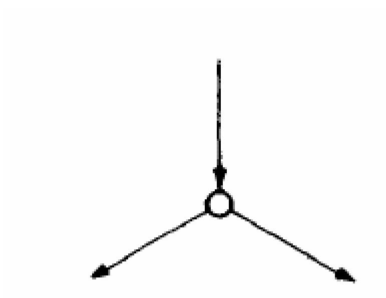
The Global Picture



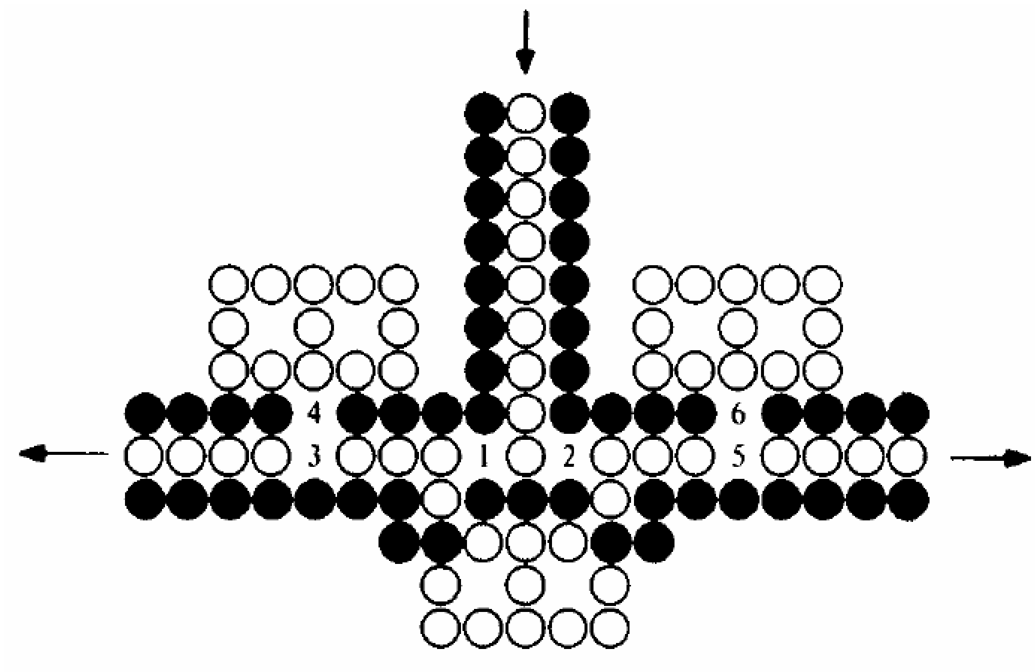
Widgets: Edges



Widget: Vertex type 1



Player A's choice



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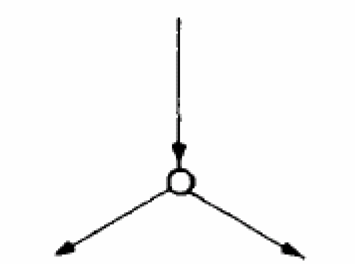
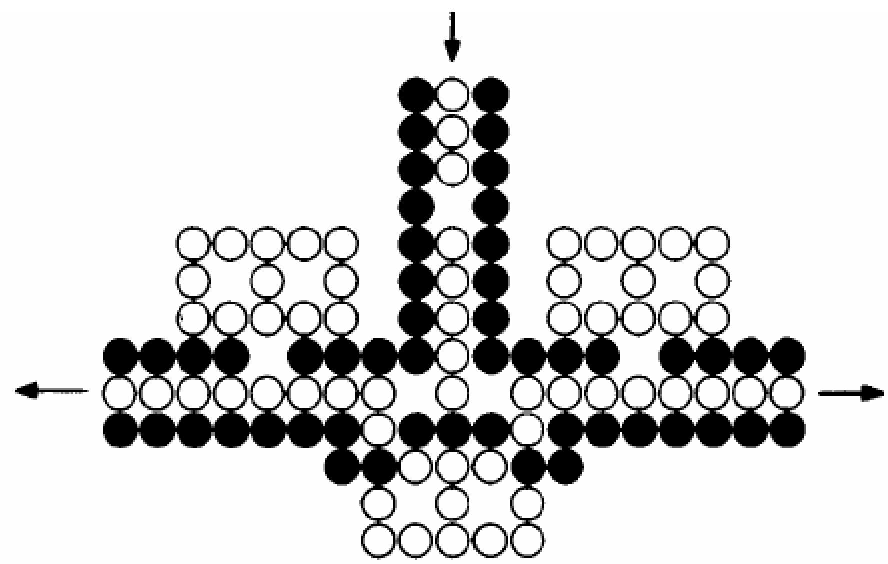
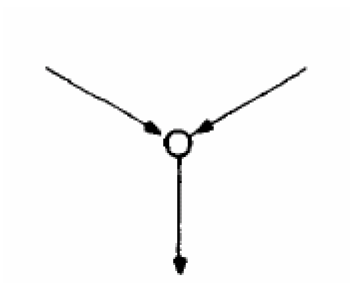


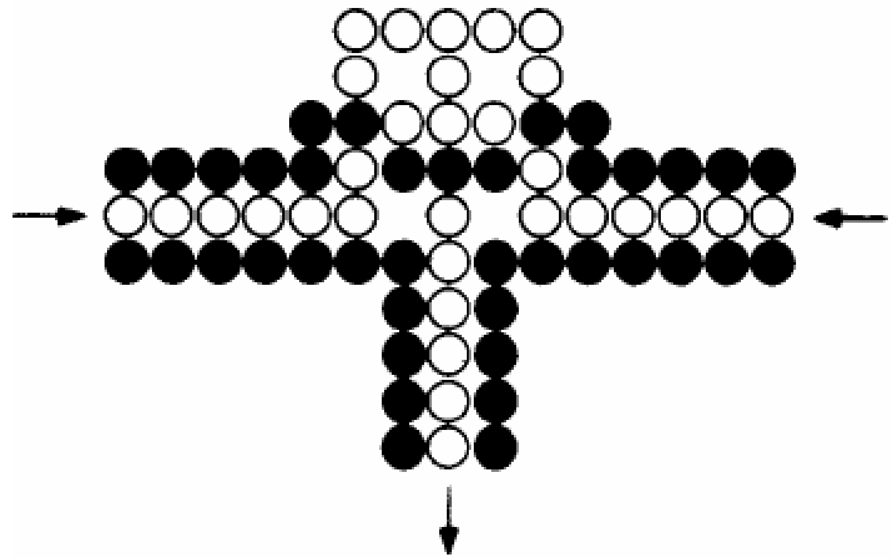
Player E's choice



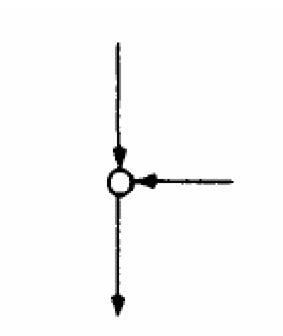
Widget: Vertex type 3



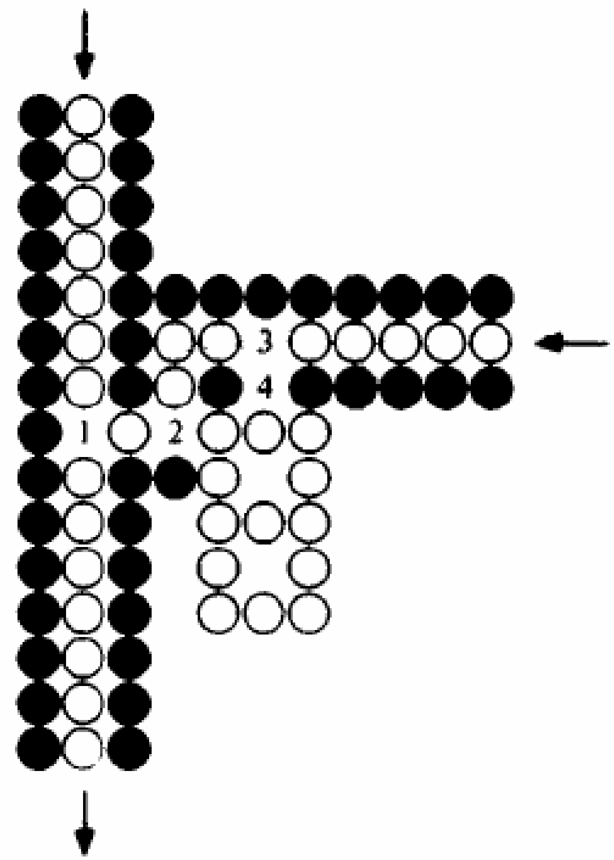
join



Widget: Vertex type 4



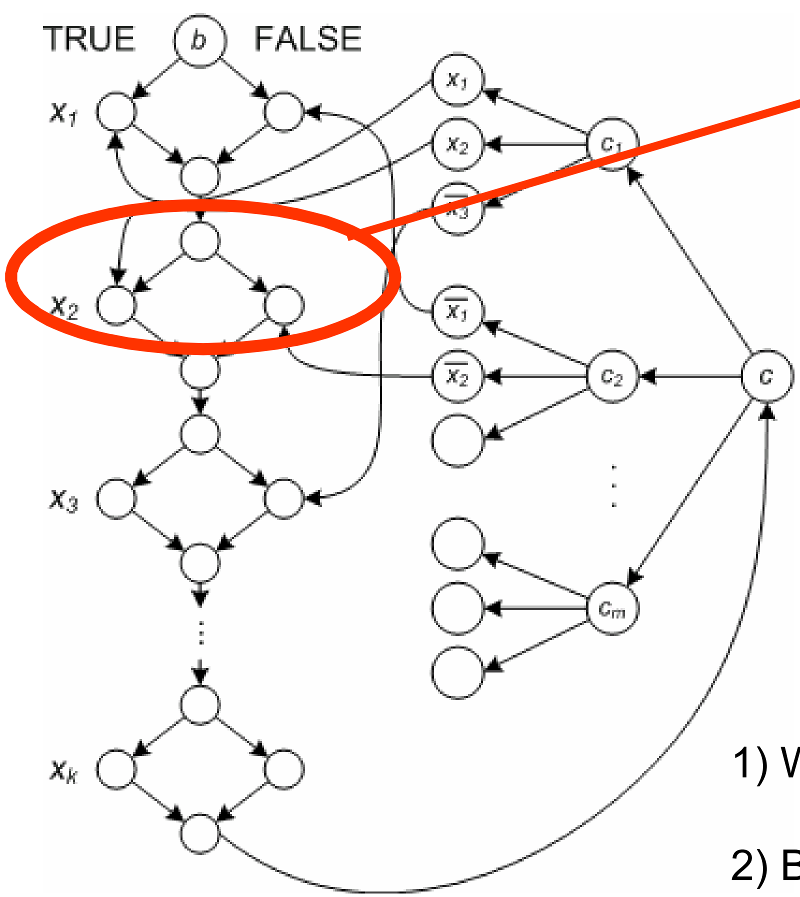
test



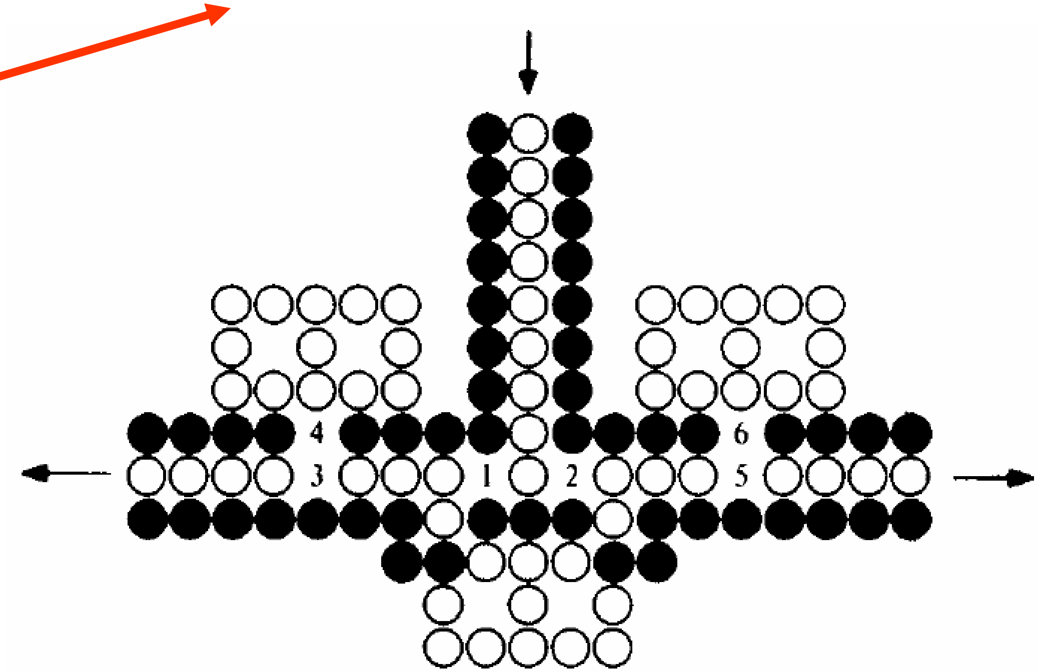
The Game Play (review)

- We connect the go configurations to make the planar GG out of Go widgets.
- White and Black make choices by playing go moves
- In the end – either white survives or is taken. If white survives – then player “A” wins (the original FQBF was not satisfiable). Else black survives.

A Close Look



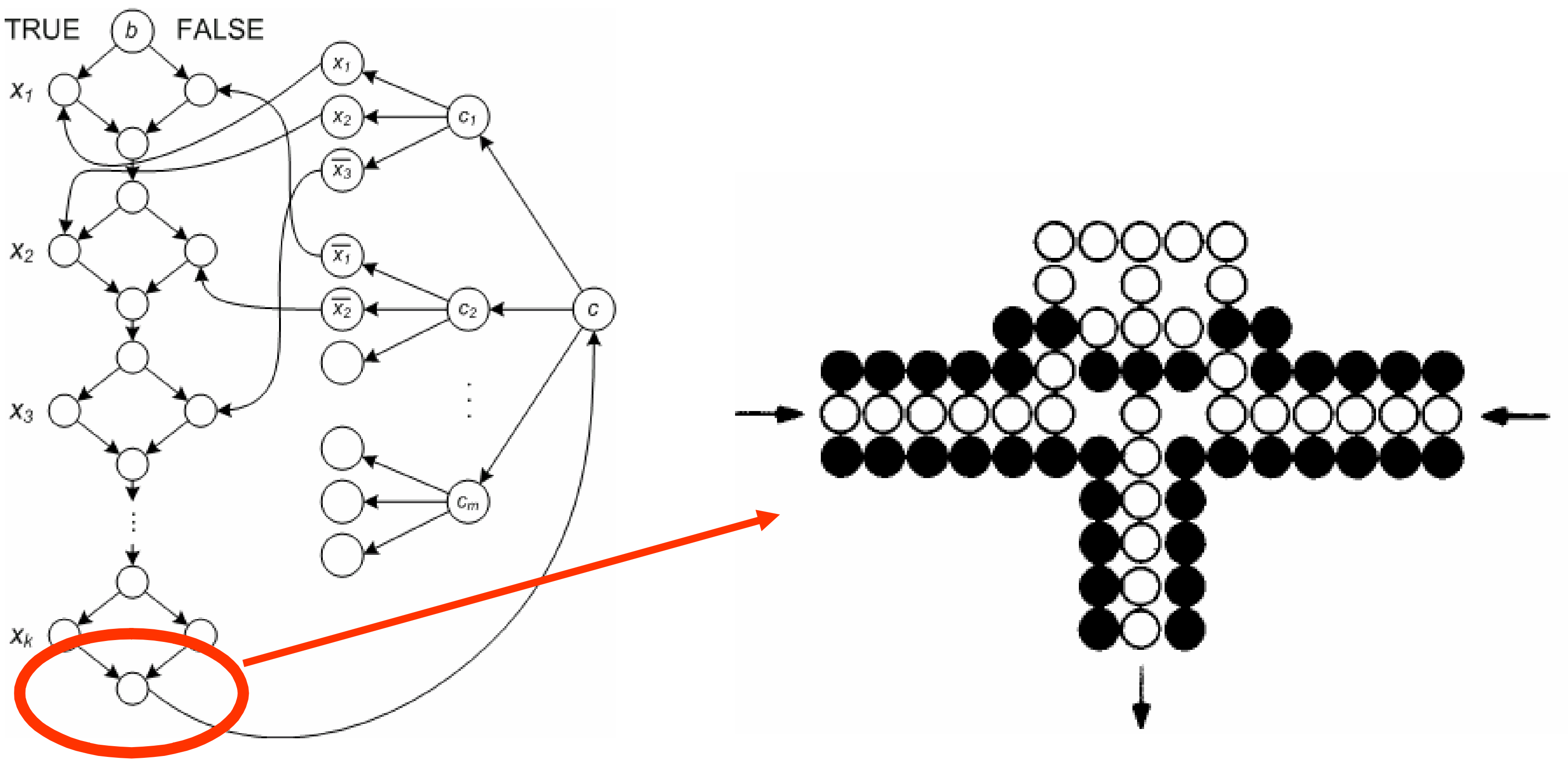
A choice by the A player



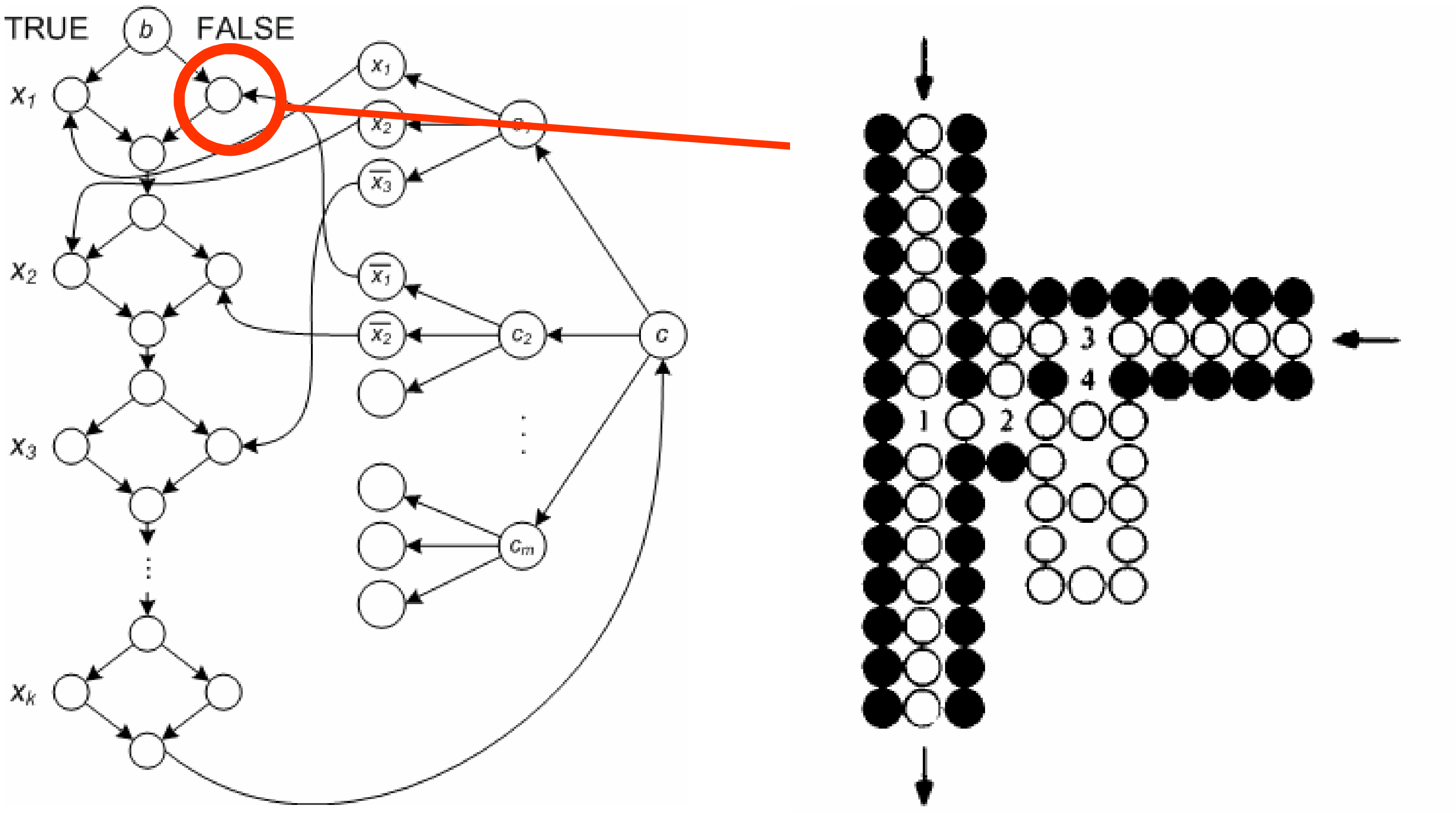
1) White must play at 1 or 2 (choosing true or false) or lose
say white played 1 (ok because of symmetry)
2) Black must respond at 2 or lose
3) White must play at 3 or lose
4) Black must respond at 4 or lose

A Close Look Continued

- The choice for player E is similar argument, only black chooses sequence.
- The choice for the “join” is obvious

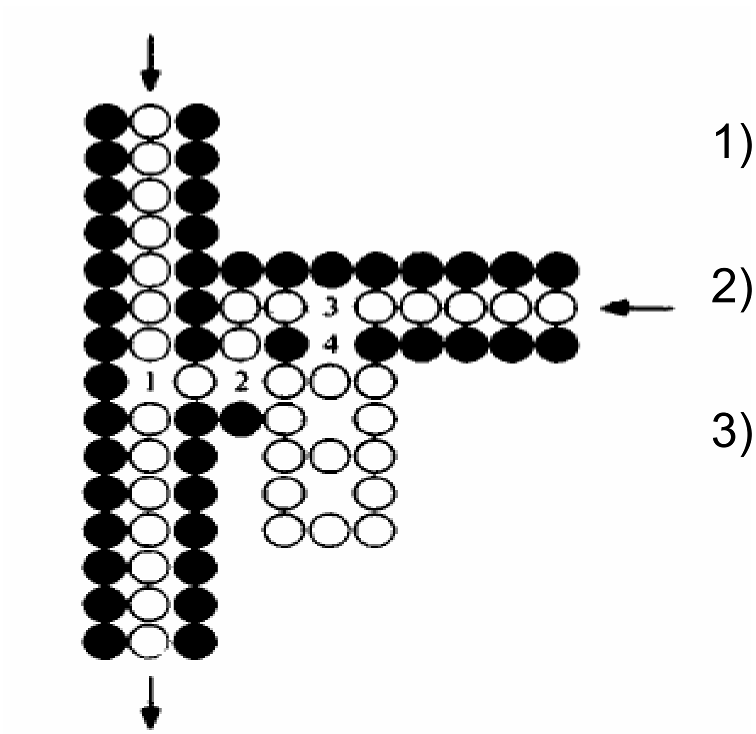


The Test



The Test

- We show that if play enters through right hand pipe, black wins iff play previously passed through the vertical pipe.



1) If play already passed through junction on top, it will have left on bottom.
2) If play subsequently enters through the right-hand pipe, Black wins.
3) If play enters through right hand pipe first, then White wins.

QED

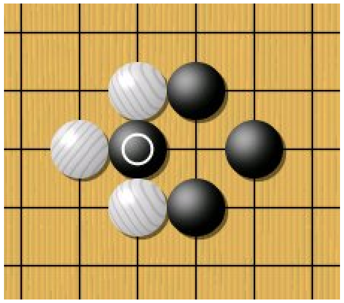
- We have reduced planar GG to Go. Black wins this game if player E would have won the GG (Which would happen only if TQBF was true), Showing Go is NP hard!
- This means that looking at a Go position, it’s (probably) hard to tell who will win. In fact, it’s PSPACE hard.
- I cheated on a little part of the proof – did anyone catch me?

Just One More Thing…

- Reduce TQBF to Generalized Geography
- Reduce Generalized Geography to Planar Generalized Geography
- Teach you the Rules of Go
- Reduce Planar Geography to Go
- Confuse you a little more

Ko, Super Ko, and EXPTIME

- There is another rule to the game of Go - called Ko. It leads to Ko-fights. Some rule sets also have Super Ko rules.



- With Ko, Go is EXPTIME-complete.
- *As far as I know*, Go (without Ko) has neither been shown to be in PSPACE nor has it been shown to be EXPTIME-complete.
- With super-Ko, what is the complexity of Go?

References

- D. Lichtenstein and M. Sipser, Go is hard, *J. ACM* 27 (1980) 393-401. polynomial-space
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- M. Crâşmaru and J. Tromp, Ladders are PSPACE-complete, Proc. 2nd Int. Conf. Computers and Games, Springer-Verlag, 2000, pp. 241-249.